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2003 J. Phys. A: Math. Gen. 36 10269

(http://iopscience.iop.org/0305-4470/36/41/002)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) 10269–10277 PII: S0305-4470(03)63482-7

A spherical Hopfield model

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Received 14 May 2003, in final form 18 August 2003
Published 1 October 2003
Online at stacks.iop.org/JPhysA/36/10269

Abstract
A spherical Hopfield-type neural network is introduced, involving neurons and patterns that are continuous variables. Both the thermodynamics and dynamics of this model are studied. In order to have a retrieval phase a quartic term is added to the Hamiltonian. The thermodynamics of the model is exactly solvable and the results are replica symmetric. A Langevin dynamics leads to a closed set of equations for the order parameters and effective correlation and response function typical for neural networks. The stationary limit corresponds to the thermodynamic results. Numerical calculations illustrate these findings.

PACS numbers: 02.50.−r, 05.20.−y, 64.60.Cn, 87.18.Sn

1. Introduction

In general, the introduction of continuous versions of discrete spin models in statistical mechanics has been fruitful. Since the work of Kac and Berlin [1], the spherical model has led to a better understanding of a lot of basic phenomena due to the fact that the mathematical calculations become considerably simpler in this continuous limit. In spin glass research, an important role has been played by the spherical p-spin spin glass model [2, 3]. Subsequently, it was quite remarkable to see how a mean-field model with spherical spins exhibits properties that resemble those of realistic structural glasses (see, e.g., the many references in [4]).

In neural network theory the Hopfield model is by now a well-known *classic* found in many textbooks (e.g., [5, 6]); it provides a prototype to study storage and retrieval of associative memories. Remarkably, a spherical version of this model has not yet been studied in the literature. One of the reasons might be that relaxing the neurons from Ising-type to spherical-type variables leads to the disappearance of the retrieval phase, i.e. one of the fundamental properties of the model is lost.

Here we show that this problem has an easy and elegant solution. By introducing a quartic potential term in the Hopfield Hamiltonian for spherical neurons, the retrieval phase is recovered. In this way we obtain a simplified version of the Hopfield model for which the thermodynamics can be exactly solved. The results are replica symmetric with marginal stability. A phase diagram is obtained; the explicit solution for the spin glass retrieval 'transition' line shows no reentrance. Moreover, the region of global stability for the retrieval solutions is larger than the corresponding region in the standard Hopfield model.

Next, we study the relaxational Langevin dynamics of this model; we obtain a closed set of equations for the order parameters and effective correlation and response functions typical for neural networks. These equations are similar to those of other models with continuous spins (see, e.g, [7] and references therein). We discuss the evolution of the overlap order parameter in the retrieval phase. The stationary limit of the dynamics is found to correspond to the thermodynamic results. Numerical solutions of the dynamical equations illustrate this behaviour.

The rest of the paper is organized as follows. In section 2 we introduce the model. In section 3 we discuss its thermodynamic properties, including the temperature–capacity phase diagram. In section 4 we study the dynamics and show the evolution of the retrieval overlap. Section 5 is a summary of our conclusions.

2. A spherical Hopfield model

The Hopfield model [8] is defined through the following mean-field Ising-type Hamiltonian:

$$\mathcal{H}(\boldsymbol{\sigma}) = -\frac{1}{2}\sum_{i,j=1}^{N} J_{ij}\sigma_i\sigma_j \tag{1}$$

where the couplings J_{ij} are related to the information to be stored in the network through the Hebbian rule

$$J_{ij} = \frac{1}{N}\sum_{\mu=1}^{p}\xi_i^\mu\xi_j^\mu \qquad J_{ii} = 0 \tag{2}$$

with $p = \alpha N$, where α is the loading capacity of the network. The N components of the p patterns ξ_i^μ are chosen to be a collection of continuous independent identical random variables (i.i.d.r.v.) with respect to i and μ, drawn from a gaussian distribution

$$P\left(\xi_i^\mu\right) = \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{\left(\xi_i^\mu\right)^2}{2}\right] \tag{3}$$

and the neurons (spins) are also taken to be continuous and to satisfy the spherical constraint

$$\sum_{i=1}^{N}\sigma_i^2 = N. \tag{4}$$

As we will show explicitly below, this formulation of the spherical Hopfield model does not allow for a retrieval phase. Therefore, we add the following term to the Hamiltonian (1):

$$-\frac{u_0}{4}\sum_{i,j,k,l} J_{ijkl}\sigma_i\sigma_j\sigma_k\sigma_l \qquad J_{ijkl} = \frac{1}{N^3}\sum_{\mu=1}^{p}\xi_i^\mu\xi_j^\mu\xi_k^\mu\xi_l^\mu \tag{5}$$

this term is quartic in the order parameter m characterizing the retrieval phase,

$$m^\mu = \frac{1}{N}\sum_{i=1}^{N}\xi_i^\mu\sigma_i. \tag{6}$$

As we discuss in the following section, the introduction of this quartic term turns out to contribute macroscopically to the condensed part of the free energy, but it only leads to sub-extensive contributions to the noise produced by the non-condensed patterns and to a sub-extensive contribution coming from the diagonal terms.

3. Thermodynamic and retrieval properties

We apply the standard replica technique [9] to the study of the thermodynamic properties of the model defined above. Starting from the Hamiltonian

$$\mathcal{H}(\boldsymbol{\sigma}) = -\frac{1}{2}\sum_{i\neq j} J_{ij}\sigma_i\sigma_j - \frac{u_0}{4}\sum_{i,j,k,l} J_{ijkl}\sigma_i\sigma_j\sigma_k\sigma_l \tag{7}$$

the partition function is given by

$$\mathcal{Z}(\beta) = \int_{-\infty}^{\infty}\left[\prod_{i=1}^{N}\mathrm{d}\sigma_i\right]\delta\left(\sum_{i=1}^{N}\sigma_i^2 - N\right)\exp[-\beta\mathcal{H}(\boldsymbol{\sigma})]. \tag{8}$$

We denote the average over the quenched disorder by $\langle\!\langle\cdots\rangle\!\rangle$ and use the replica method to compute the free energy per site f

$$\beta N f = -\langle\!\langle \ln \mathcal{Z}(\beta)\rangle\!\rangle = -\lim_{n\to 0}\frac{1}{n}\langle\!\langle \mathcal{Z}^n(\beta) - 1\rangle\!\rangle \tag{9}$$

where the replicated partition function $\mathcal{Z}^n(\beta)$ is given by

$$\mathcal{Z}^n(\beta) = \int_{-\infty}^{\infty}\left[\prod_{i=1}^{N}\prod_{\alpha=1}^{n}\mathrm{d}\sigma_i^\alpha\right]\prod_{\alpha=1}^{n}\delta\left(\sum_{i=1}^{N}\left(\sigma_i^\alpha\right)^2 - N\right)\exp\left[-\beta\sum_{\alpha=1}^{n}\mathcal{H}\left(\left\{\sigma_i^\alpha\right\}\right)\right]. \tag{10}$$

We assume, for simplicity, one condensed pattern, say the first one, meaning that the retrieval overlap order parameter defined as

$$m_\alpha^\mu = \frac{1}{N}\sum_{i=1}^{N}\xi_i^\mu\sigma_i^\alpha \tag{11}$$

is of order $\mathcal{O}(1)$ for pattern $\mu = 1$ and of order $\mathcal{O}(1/\sqrt{N})$ for the other patterns $\mu \geqslant 2$. We then introduce this order parameter in the partition function through a δ-function and split the partition function into a condensed and a non-condensed part. Next, the disorder average has to be performed. The average over the non-condensed patterns can be done by introducing the neuron (spin) overlap order parameter

$$q_{\alpha\beta} = \frac{1}{N}\sum_{i=1}^{N}\sigma_i^\alpha\sigma_i^\beta \qquad \alpha\neq\beta \tag{12}$$

into the partition function through a δ-function representation and performing as many integrals as possible. Furthermore, since the patterns are gaussian the average over the condensed patterns can be done exactly without destroying the gaussian nature of the model. Finally, also the integral over the spins can be worked out leading to the following expression for the replicated free energy per site:

$$\beta f = \lim_{N\to\infty}\lim_{n\to 0}\frac{-1}{Nn}\int_{-\infty}^{\infty}\prod_{\alpha=1}^{n}\mathrm{d}m_\alpha\int_{-\infty}^{\infty}\prod_{\alpha\neq\beta}\mathrm{d}q_{\alpha\beta}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\prod_{\alpha=1}^{n}\frac{\mathrm{d}u_\alpha}{4\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\prod_{\alpha\neq\beta}\frac{\mathrm{d}\hat{q}_{\alpha\beta}}{4\pi\mathrm{i}/N}\,\mathrm{e}^{-Ng} \tag{13}$$

with

$$g = -\frac{\beta}{2}\sum_{\alpha=1}^{n}m_\alpha^2 - \frac{u_0\beta}{4}\sum_{\alpha=1}^{n}m_\alpha^4 + \frac{n\alpha\beta}{2} + \frac{1}{2}\sum_{\alpha,\beta=1}^{n}m_\alpha(q^{-1})_{\alpha\beta}m_\beta - \alpha n\ln 2\pi$$
$$+\frac{\alpha}{2}\ln\det(\mathbb{1} - \beta\mathbf{q}) - \frac{n}{2}\ln 2\pi + \frac{1}{2}\ln\det\widehat{\mathbf{Q}} - \frac{1}{2}\sum_{\alpha,\beta=1}^{n}\widehat{Q}_{\alpha\beta}q_{\alpha\beta}. \tag{14}$$

In this expression we have denoted $m^1_\alpha = m_\alpha$, the $\hat{q}_{\alpha\beta}$ are conjugate variables and

$$\widehat{Q}_{\alpha\beta} = u_\alpha \delta_{\alpha\beta} - \hat{q}_{\alpha\beta}(1-\delta_{\alpha\beta}). \tag{15}$$

Within a *K*th order Parisi replica symmetry-breaking scheme, we assume that the spin overlap matrix has the following ultrametric structure:

$$q_{\alpha\beta} = q_i \qquad \text{if} \quad I(\alpha/m_i) \neq I(\beta/m_i) \quad \text{and} \quad I(\alpha/m_{i+1}) = I(\beta/m_{i+1}) \tag{16}$$

with $\{q_i\}_{i=0,\ldots,K}$ a set of real numbers and $\{m_i\}_{i=1,\ldots,K}$ a set of integers such that m_{i+1}/m_i is an integer ($m_0 = n,\ m_{K+1} = 1$). We introduce the inverse of the Parisi function

$$x(q) = n + \sum_{i=0}^{K} (m_{i+1} - m_i)\,\Theta\,(q - q_i). \tag{17}$$

We consider the limit $K \to \infty, q_0 \to 0$ and $q_K \to q_M$, such that the free energy per site becomes, in the limit $n \to 0$, $N \to \infty$,

$$\begin{aligned}\beta f[x(q), m] = &-\frac{\beta}{2}m^2 - \frac{u_0\beta}{4}m^4 + \frac{\alpha\beta}{2} - \left(\alpha + \frac{1}{2}\right)\ln 2\pi - \frac{1}{2} \\ &+ \frac{\alpha}{2}\left[\ln(1-\beta(1-q_M)) - \beta\int_0^{q_M} \frac{\mathrm{d}q}{1-\beta\int_q^1 x(q')\,\mathrm{d}q'}\right] \\ &- \frac{1}{2}\left[\int_0^{q_M} \frac{\mathrm{d}q}{\int_q^1 x(q')\,\mathrm{d}q'} + \ln(1-q_M) - \frac{m^2}{\int_0^1 x(q)\,\mathrm{d}q}\right]\end{aligned} \tag{18}$$

where f has to be extremized with respect to $x(q)$ and m. Assuming without loss of generality that $x(q)$ is a piecewise continuous function, it is straightforward to show [2, 10] that the solution for $x(q)$ is of the form

$$x(q) = \Theta(q - q_M). \tag{19}$$

Hence, the replica symmetric solution is exact. The free energy is given by

$$\begin{aligned}\beta f = \text{extr}\Bigg\{&-\frac{\beta}{2}m^2 - \frac{u_0\beta}{4}m^4 + \frac{\alpha\beta}{2} - \left(\alpha + \frac{1}{2}\right)\ln 2\pi - \frac{1}{2} \\ &+ \frac{\alpha}{2}\left[\ln(1-\beta(1-q)) - \frac{\beta q}{1-\beta(1-q)}\right] - \frac{1}{2}\left[\ln(1-q) + \frac{q-m^2}{1-q}\right]\Bigg\}\end{aligned} \tag{20}$$

with $q_M = q$, the Edwards–Anderson order parameter.

The solution is validated by investigating its stability against replica symmetry-breaking fluctuations, to find that the replicon eigenvalue is zero and that the replica symmetric result is therefore marginally stable. This is done by studying the Hessian matrix formed by the coefficients in an expansion of the free energy to second order, in small deviations of the order parameters $q_{\alpha\beta}$ and m_α from their replica symmetric values. Stability requires that this matrix should be at least positive semi-definite. Three classes of eigenvectors, respectively, those invariant under permutation of all replicas, all replicas but 1 and all replicas but 2, and their corresponding eigenvalues are found. Positivity of two of these eigenvalues is equivalent to the saddle-point stability condition of the replica symmetric free energy with respect to q and m. The condition for positivity of the third eigenvalue, called the replicon eigenvalue, indicating stability with respect to replica symmetry breaking is given by

$$\lambda_R = \frac{\partial^2 f}{\partial q_{\alpha\beta}\partial q_{\alpha\beta}} - 2\frac{\partial^2 f}{\partial q_{\alpha\beta}\partial q_{\alpha\gamma}} + \frac{\partial^2 f}{\partial q_{\alpha\beta}\partial q_{\gamma\delta}} \geqslant 0. \tag{21}$$

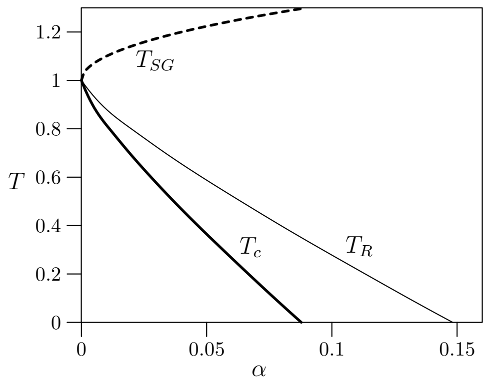

Figure 1. T–α phase diagram for the spherical Hopfield model. Full (dashed) lines indicate discontinuous (continuous) transitions: T_{SG} describes the spin glass transition and T_R ((24)–(27)) indicates the boundary for the existence of retrieval solutions, T_c denotes the thermodynamic transition below which the retrieval states are global minima of the free energy.

For the spherical Hopfield model the explicit calculation of these derivatives results in $\lambda_R = 0$, showing the marginal stability of the solution.

The saddle-point equations become simple algebraic equations for q and m:

$$m\left(u_0 m^2 - \frac{1-\chi}{\chi}\right) = 0 \qquad \frac{q-m^2}{\chi^2} = \frac{\alpha q}{(1-\chi)^2} \tag{22}$$

with susceptibility $\chi = \beta(1-q)$. Note that if we remove the quartic interaction by setting $u_0 = 0$, there is no solution with $m \neq 0$ and there is no retrieval phase.

We therefore set $u_0 = 1$. The corresponding T–α phase diagram is shown in figure 1. This result is compared with the phase diagram of the standard Hopfield model calculated in a replica symmetric approximation [5, 11]. Again, there are three phases. For temperatures above the broken line at T_{SG}, there exist paramagnetic solutions characterized by $m = q = 0$; below the broken line, spin glass solutions with $m = 0$ but $q \neq 0$, exist. The transition between the paramagnetic and the spin glass phase is continuous, and the line T_{SG} separating the two phases is easily computed:

$$T_{\mathrm{SG}}(\alpha) = 1 + \sqrt{\alpha} \tag{23}$$

for $0 \leqslant \alpha < \infty$, as in the standard Hopfield model. Below the thin full line T_R, retrieval solutions with $m \neq 0$ and $q \neq 0$ appear. The transition between the spin glass and the retrieval phase is discontinous. The fact that the values of both q and m jump when passing through this line leads to an analytical expression for T_R. Equations (22) can be written as two polynomials in q and m; a jump means that complex roots of these two polynomials become real at the same time for all values of $(\alpha, T_R(\alpha))$. After some algebra, this condition leads to

$$\alpha_{R\text{-}SG}(\beta) = \frac{2}{81\beta^5}\left[8(\beta-1)^2\beta^2(7\beta-16) + \mathrm{sgn}(\Delta)\frac{\Sigma}{|\Delta|^{1/3}} + \mathrm{sgn}(\Delta)|\Delta|^{1/3}\right] \tag{24}$$

with

$$\Delta = 81\sqrt{3}(\beta-1)^{9/2}\beta^{17/2}(|125\beta-128|)^{3/2} - \Delta_2 \tag{25}$$

$$\Delta_2 = (\beta-1)^5\beta^6\{-2097\,152 + \beta(4849\,664 + \beta[-3459\,072 + 625\beta(1088 + 25\beta)])\} \tag{26}$$

$$\Sigma = (\beta-1)^3\beta^4\{-16\,384 + 5\beta[6144 + \beta(-2976 + 125\beta)]\}. \tag{27}$$

At $T = 0$, the line T_R defines the critical capacity $\alpha(T = 0) = 4/27 = 0.148\,148\,15$, as seen in figure 1.

Below the thick full line T_c, the retrieval states become globally stable. Note that, in contrast to the standard Hopfield model, there is no reentrant behaviour in the transitions, since the spherical model is (marginally) replica symmetric. Furthermore, the region of global stability for the retrieval solutions is much larger in the spherical model.

4. Macroscopic dynamics

In this section we study the dynamics of the model. Since the neurons are continuous, we assume relaxational dynamics given by the Langevin equations

$$\frac{\partial \sigma_i(t)}{\partial t} = -\mu(t)\sigma_i(t) - \frac{\delta \mathcal{H}(\boldsymbol{\sigma})}{\delta \sigma_i(t)} + \eta_i(t) + \theta_i(t) \tag{28}$$

where the first term on the rhs controls the fluctuations of the neurons, $\eta_i(t)$ is gaussian noise with moments

$$\langle \eta_i(t) \rangle = 0 \qquad \langle \eta_i(t)\eta_j(t') \rangle = 2T\delta_{ij}\delta(t - t') \tag{29}$$

and $\theta_i(t)$ is an external perturbation field. We investigate the dynamical properties of this system using the generating functional approach [12, 13]. It is an exact procedure based on finding a microscopic path in time. The statistics of paths are fully captured by the following moment generating function:

$$\mathcal{Z}[\psi] = \int \mathbf{D}[\boldsymbol{\sigma}, \hat{\boldsymbol{\sigma}}] \exp\left[\sum_{i=1}^{N} \int \mathrm{d}t\ \psi_i(t)\,\mathrm{i}\sigma_i(t) + A[\boldsymbol{\sigma}, \hat{\boldsymbol{\sigma}}]\right] \tag{30}$$

with action $A[\boldsymbol{\sigma}, \hat{\boldsymbol{\sigma}}]$ given by

$$A[\boldsymbol{\sigma}, \hat{\boldsymbol{\sigma}}] = \sum_{i=1}^{N} \int \mathrm{d}t \left\{ -T\hat{\sigma}_i^2(t) + \mathrm{i}\hat{\sigma}_i(t) \left(\frac{\partial \sigma_i(t)}{\partial t} + \mu(t)\sigma_i(t) - \theta_i(t) + \frac{\delta \mathcal{H}(\boldsymbol{\sigma})}{\delta \sigma_i(t)} \right) \right\} \tag{31}$$

where the $\psi_i(t)$ are the generating fields, $\hat{\sigma}_i(t)$ are conjugate variables and $\mathbf{D}[\boldsymbol{\sigma}, \hat{\boldsymbol{\sigma}}]$ is the measure in path space. From this expression, all physical quantities of interest can be computed by taking derivatives with respect to the generating and the external perturbation fields. The path average of the spin, the correlation functions and the response functions are given by

$$\langle \sigma_i(t) \rangle = -\mathrm{i} \lim_{\psi \to 0} \frac{\delta \mathcal{Z}[\psi]}{\delta \psi_i(t)} \tag{32}$$

$$C_{ij}(t, t') = -\lim_{\psi \to 0} \frac{\delta^2 \mathcal{Z}[\psi]}{\delta \psi_i(t)\delta \psi_j(t')} \tag{33}$$

$$G_{ij}(t, t') = -\mathrm{i} \lim_{\psi \to 0} \frac{\delta^2 \mathcal{Z}[\psi]}{\delta \theta_j(t')\delta \psi_i(t)}. \tag{34}$$

The idea is now that for $N \to \infty$, only the statistical properties of the stored patterns influence the macroscopic quantities, such that the generating functional can be averaged over all pattern realizations, i.e., over the disorder.

The averaging over the disorder can be done by introducing, e.g., through appropriate δ-function representations, the retrieval overlap $m(t)$ and spin overlap $q(t, t')$

$$m(t) = \frac{1}{N} \sum_{i=1}^{N} \xi_i^1 \sigma_i(t) \qquad q(t, t') = \frac{1}{N} \sum_{i=1}^{N} \sigma_i(t)\sigma_i(t') \tag{35}$$

analogous order parameters for the conjugate neuron variables, and order parameters of a mixed type

$$g(t,t') = \frac{1}{N}\sum_{i=1}^{N}\sigma_i(t)\hat{\sigma}_i(t'). \tag{36}$$

Here we have again assumed that only the first pattern is condensed. After a certain amount of algebra one finds that one can achieve site factorization in the formula and hence the dynamics can be described in terms of a single effective neuron with an effective local field depending on past states of the neuron and on an effective coloured noise.

The effective single-site dynamics in the thermodynamic limit is given by

$$\frac{\partial\sigma(t)}{\partial t} = -\mu(t)\sigma(t) + [m(t) + u_0 m^3(t)]\xi^1 + \theta(t) + \alpha\int \mathrm{d}t'\,[(\mathbb{1}-\mathbf{G})^{-1}\mathbf{G}](t,t')\sigma(t') + \phi(t) \tag{37}$$

where ϕ is coloured noise. Let us denote by $\langle\langle\cdots\rangle\rangle_{\xi^1}$ the average over the condensed pattern, and by $\langle\cdots\rangle_\star$ the average over the effective path defined by (37) and over the noise ϕ. Then, the mean value and variance of ϕ are

$$\langle\phi(t)\rangle_\star = 0 \qquad \langle\phi(t)\phi(t')\rangle_\star = 2T\delta(t-t') + \alpha[\mathbb{1}-\mathbf{G}]^{-1}\mathbf{C}[\mathbb{1}-\mathbf{G}^\dagger]^{-1}(t,t'). \tag{38}$$

Furthermore, the retrieval overlap $m(t)$ is given by

$$m(t) = \langle\langle\xi^1\langle\sigma(t)\rangle_\star\rangle\rangle_{\xi^1} \tag{39}$$

and the matrices $\mathbf{C}$ and $\mathbf{G}$ are the dynamical order parameters of the problem, i.e. the correlation and response functions

$$C(t,t') = \langle\langle\langle\sigma(t)\sigma(t')\rangle_\star\rangle\rangle_{\xi^1} \qquad G(t,t') = \frac{\partial}{\partial\theta(t')}\langle\langle\langle\sigma(t)\rangle_\star\rangle\rangle_{\xi^1}. \tag{40}$$

The latter are precisely the order parameters $q(t,t')$ and $\mathrm{i}g(t,t')$ introduced above, at the saddle point. We remark that the calculation reveals explicitly that the quartic term in the Hamiltonian gives no extensive noise contribution.

Taking the perturbation field $\theta(t)$ to zero, and using the causality properties of the response function and the spherical constraint $C(t,t) = 1$, it is possible to write a closed set of equations for the macroscopic observables (39) and (40) that specify the dynamics

$$\left(\frac{\partial}{\partial t} + \mu(t)\right)m(t) = [m(t) + u_0 m^3(t)] + \alpha\int_{-\infty}^{t}\mathrm{d}t'\,R(t,t')m(t') \tag{41}$$

$$\left(\frac{\partial}{\partial t} + \mu(t)\right)G(t,t') = \delta(t-t') + \alpha\int_{t'}^{t}\mathrm{d}t_1\,R(t,t_1)G(t_1,t') \tag{42}$$

$$\left(\frac{\partial}{\partial t} + \mu(t)\right)C(t,t') = 2TG(t',t) + \alpha\int_{-\infty}^{t'}\mathrm{d}t_1\,S(t,t_1)G(t',t_1) + [m(t) + u_0 m^3(t)]m(t') + \alpha\int_{-\infty}^{t}\mathrm{d}t_1\,R(t,t_1)C(t',t_1) \tag{43}$$

where we have defined the effective correlation function $S(t,t')$ and the response function $R(t,t')$ as

$$S(t,t') = [\mathbb{1}-\mathbf{G}]^{-1}\mathbf{C}[\mathbb{1}-\mathbf{G}^\dagger]^{-1}(t,t') \qquad R(t,t') = [(\mathbb{1}-\mathbf{G})^{-1}\mathbf{G}](t,t'). \tag{44}$$

In order to obtain the stationary state from equations (41)–(43) we assume that close to equilibrium the order parameters become time translationally invariant: the one-time quantities

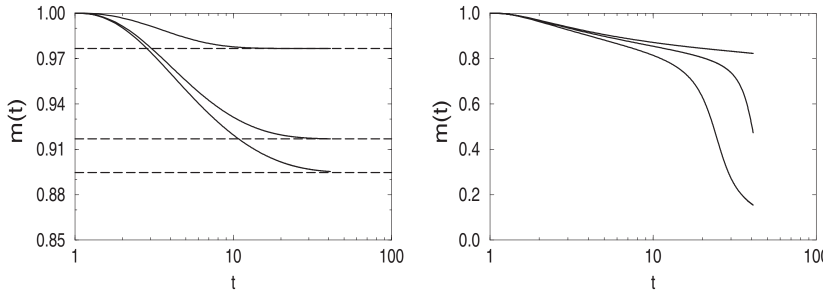

Figure 2. The overlap order parameter $m(t)$ as a function of time at temperature $T = 0$ for several values of the capacity: $\alpha = 0.041, 0.112, 0.127$ (from top to bottom) on the left and $\alpha = 0.150, 0.160, 0.180$ (from top to bottom) on the right. The horizontal dashed lines correspond to the stationary values from the thermodynamic theory.

become time independent and the two-time quantities satisfy $C(t,t') = C(t-t')$, $G(t,t') = G(t-t')$ (and similarly for R and S). Time translational invariance holds when the system is ergodic; the correlation and response functions are then related through the fluctuation–dissipation theorem (FDT) [3, 4]

$$\beta\partial_\tau C(\tau) = G(-\tau) - G(\tau) \qquad \beta\partial_\tau S(\tau) = R(-\tau) - R(\tau) \tag{45}$$

with $\tau = t - t'$, the initial time $t' = -\infty$ and $\partial/\partial\tau$ denoted by ∂_τ. These assumptions lead to the following evolution equation for the correlation function:

$$\begin{aligned}(\partial_\tau + \mu - \alpha\beta[1 - S(\tau)])C(\tau) + \alpha\beta \int_0^\tau \mathrm{d}t'\, [S(\tau - t') - S(\tau)]\partial_{t'} C(t') \\ = [m + u_0 m^3]m + \alpha \int_0^\infty \mathrm{d}t'\, [R(t'+\tau)C(t') + S(t'+\tau)G(t')].\end{aligned} \tag{46}$$

The conditions $C(0) = 1$ and $\beta\partial_\tau C(\tau)|_{\tau=0} = -1$ lead to

$$[m + u_0 m^3]m = \frac{C(\infty)}{\beta[1 - C(\infty)]} - \alpha\beta S(\infty)[1 - C(\infty)] \tag{47}$$

for $\tau \to \infty$. Finally, using the evolution equation for m and

$$\lim_{\tau\to\infty} C(\tau) = q \qquad \lim_{\tau\to\infty} S(\tau) = \frac{q}{[1 - \beta(1-q)]^2} \tag{48}$$

it is straightforward to arrive at the equilibrium saddle-point equations (22).

If we follow the argumentation in [3] by starting from the evolution equation for the correlation function, written as

$$(\partial_\tau + \mu - \alpha\beta[1 - S(\tau)])\beta[1 - C(\tau)] - \alpha\beta^2 \int_0^\tau \mathrm{d}t'[S(\tau - t') - S(\tau)]\partial_{t'} C(t') = 1 \tag{49}$$

and use the fact that the dynamics is purely relaxational so that $\partial_t C(t) \leqslant 0$, $\partial_t S(t) \leqslant 0$, we can derive the following condition for ergodicity:

$$\frac{1}{\beta[1 - C(\tau)]} + \alpha\beta - \mu - \mu S(\tau) \geqslant 0. \tag{50}$$

This inequality seems to indicate that our system is ergodic in the sense of [3].

In figure 2 we illustrate the dynamical behaviour that follows from discretizing equations (41)–(43). Specifically, the overlap order parameter is shown as a function of time for several values of the loading capacity at zero temperature, with $m(0) = 1$. The behaviour for non-zero temperature is qualitatively the same. We clearly see that below the critical capacity (figure on the left) the system evolves to the stationary solution, while above the critical capacity $\alpha_c = 0.148\,148\,15$ (figure on the right) $m(t)$ drops to zero.

5. Conclusions

We have presented the spherical version of the Hopfield model. A quartic interaction that does not destroy the spherical character of the model has been introduced in order to sustain a retrieval phase. The thermodynamic phase diagram is qualitatively similar to that for the standard Hopfield model, except that there is no reentrance because the system is (marginally) replica symmetric. Furthermore, the region of global stability for the retrieval solutions is larger than in the discrete Hopfield model. A closed set of equations is obtained for the Langevin dynamics, and the stationary limit is shown to correspond to the thermodynamic results. A numerical calculation of the evolution of the retrieval overlap order parameter illustrates these findings.

Acknowledgments

We thank A C C Coolen and R Kühn for informative discussions. We are grateful to the referee for some constructive remarks on the text. This work has been supported in part by the Fund of Scientific Research, Flanders-Belgium.

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